Three-Dimensional Elliptical Centered MHD Equilibria *

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MHD equilibria are considered which are purely elliptical and centered with respect to the magnetic axis up to third order in the distance from the magnetic axis. Their β-value is not governed by conventional estimates. In particular, three-dimensional, toroidal, medium aspect ratio ($A \sim 20$), medium β ($\sim 20\%$) equilibria without longitudinal current of this type are constructed.

1. Introduction

Conceptually simple toroidal equilibria, such as tokamaks and $l=2$ stellarators, are governed by the fact that an increase of the pressure gradient for fixed rotational transform, i.e. an increase in the poloidal β-value, tends to increase the magnitude of the transverse fields and to move the stagnation points of the flux surfaces towards the magnetic axis [1, 2, 3, 4]. The concept of the flux conserving tokamak (see, for example, [5]) circumvents this difficulty, but if one considers the equilibrium on the level of resistive MHD, this is at the cost of a punctilious choice of the pressure and current profiles. In this paper, we consider the question whether it is possible to choose the gross form of the plasma column in such a way that sizeable β values may be reached without deformation of the outer flux surfaces. The formal procedure is as follows. In an expansion around the magnetic axis the shifts of the magnetic surfaces with respect to the magnetic axis and their triangular deformation are described by four functions of the arc length along the magnetic axis. We construct equilibria for which these four functions vanish identically. The magnetic surfaces are then purely elliptical and centered with respect to the magnetic axis up to third order in the distance from the axis. Since there are no stagnation points in the third order flux surfaces, the conventional stagnation point discussion and equilibrium β estimate [1, 3] fails: within the framework of the conventional theory the equilibrium-β becomes unlimited and has to be determined by other considerations limiting the attainable aspect ratio.

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2. Notation and Simple Examples

The conditions for the third order functions describing the shifts and the triangularities to vanish identically are (see, for example, [3])

$$2\varkappa \sin\alpha \left[-K_0' + \alpha' \left(3e - \frac{2}{e}\right) + 4\tau e \right] + \varkappa \cos\alpha \left[\frac{c_0'}{c_0}\left(e - \frac{2}{e}\right) - \frac{e'}{e}\left(3e + \frac{2}{e}\right)\right] + 2\varkappa' \cos\alpha \left(e + \frac{2}{e}\right) + 8\pi Q_0 \dot{p}\, c_0^{1/2} e^{1/2} B_1 = 0\,, \tag{1}$$

$$2\varkappa \cos\alpha \left[K_0' + \alpha' \left(2e - \frac{3}{e}\right) - 4\tau/e \right] + \varkappa \sin\alpha \left[\frac{c_0'}{c_0}\left(\frac{1}{e} - 2e\right) + \frac{e'}{e}\left(2e + \frac{3}{e}\right)\right] + 2\varkappa' \sin\alpha \left(2e + \frac{1}{e}\right) - 8\pi Q_0 \dot{p}\, c_0^{1/2} e^{-1/2} B_2 = 0\,. \tag{2}$$

Here, $\varkappa$ and τ are curvature and torsion of the magnetic axis, $' = \mathrm{d}/\mathrm{d}l$, where l is the arc length along the magnetic axis, e is the half-axis ratio of the elliptical plasma cross-section, whose angle with respect to the normal of the magnetic axis is described by α, c_0 is the magnetic field on axis, and

$$K_0' = \left(\frac{j}{c_0} + 2\tau + 2\alpha'\right) \Big/ \left(e + \frac{1}{e}\right),$$

where j is the current density on axis. The quantity Q_0 is given by

$$Q_0 = \int \mathrm{d}l / c_0\,,$$

and $\dot{p}$ is the derivative of the pressure with respect to the volume inside the magnetic surfaces. B_1 and B_2 are given by

$$B_1 = b_\mathrm{r} \cos\alpha + b_\mathrm{i} \sin\alpha\,,$$

$$B_2 = b_\mathrm{i} \cos\alpha - b_\mathrm{r} \sin\alpha\,,$$

where $b = b_r + i\, b_i$ solves

$$b' + i(K_0' - \alpha')\, b = -\exp(i\,\alpha)\, c_0^{-3/2} \varkappa (e^{-1/2} \cos\alpha - i\, e^{1/2} \sin\alpha)\,. \tag{3}$$

We now consider four examples: axial symmetry, the $l=2$ stellarator, helical equilibria, and an analytical expansion around a flat closed curve as magnetic axis.

In axial symmetry ($\varkappa' = \tau = e' = \alpha' = c_0' = 0$) Eqs. (1), (2) cannot be solved for a decreasing pressure because they reduce to

$$\varkappa \sin\alpha \left(-K_0'^2 + \frac{4\pi Q_0}{c_0} \dot{p}\, e \right) = 0\,,$$

$$\varkappa \cos\alpha \left(K_0'^2 - \frac{4\pi Q_0}{c_0} \frac{\dot{p}}{e} \right) = 0\,.$$

For the $l=2$ stellarator with circular magnetic axis ($\tau = e' = \varkappa' = c_0' = 0$, $e \neq 1$, $\alpha' = n/(2R)$, where n is the number of field periods and R the major radius) Eqs. (1), (2) yield

$$-K_0' + \alpha' \left(3e - \frac{2}{e} \right) + \frac{4\pi Q_0}{c_0} \dot{p} \frac{K_0' e + \alpha'}{K_0'^2 - \alpha'^2} = 0\,,$$

$$K_0' + \alpha' \left(2e - \frac{3}{e} \right) - \frac{4\pi Q_0}{c_0} \dot{p} \frac{K_0'/e + \alpha'}{K_0'^2 - \alpha'^2} = 0\,.$$

For $\dot{p} < 0$ these equations are equivalent to

$$\frac{K_0'}{\alpha'} = -E + \sqrt{E^2 - 5}\,,$$

$$-\frac{2\pi \dot{p} Q_0}{c_0 \alpha'^2} = -2E + 3\sqrt{E^2 - 5}\,,$$

where $E = e + 1/e$. Thus it is seen that no solution exists for (i) $j = 0$ and (ii) $\dot{p} < 0$, $E < 3$.

Equations (1), (2) can easily be satisfied in helical symmetry for decreasing pressure. With $\varkappa' = e' = c_0' = \alpha = b_r = 0$, Eq. (1) is trivially satisfied and Eq. (2) yields

$$K_0' \left(K_0' - \frac{4\tau}{e} \right) - \frac{4\pi Q_0 \dot{p}}{c_0 e} = 0$$

or, equivalently

$$\frac{e}{(e^2+1)^2} \left[\left(\frac{j}{c_0} \right)^2 e^2 - 4\tau \frac{j}{c_0} - 4\tau^2 (e^2 + 2) \right] - \frac{4\pi Q_0 \dot{p}}{c_0} = 0\,. \tag{4}$$

In particular, this equation has solutions for $\dot{p} < 0$ and vanishing current density on the magnetic axis.

We further consider an analytical expansion around a flat closed curve as zeroth order magnetic axis. Equations (1), (2) have the following class of solutions, which we take as zeroth order

$$\varkappa_0^2 = \text{const}\; e_0 (e_0^2 + 2)\,,$$

$$\alpha_0 = \tau_0 = c_{00}' = j_0 = \dot{p}_0 = 0\,.$$

Keeping

$$c_0' = j_0 = \dot{p} \equiv 0\,,$$

we see that Eqs. (1), (2) allow the following perturbation:

$$\varkappa = \varkappa_0 + O(\varepsilon^2)\,, \quad e = e_0 + O(\varepsilon^2)\,,$$

$$\tau = \varepsilon\, \tau_1\,,$$

$$\alpha = \varepsilon\, \alpha_1\,,$$

where α_1 satisfies the equation

$$\alpha_1' = -\frac{1}{2x} \frac{x+1}{x-1} \frac{4x^2 + 7x + 4}{(2x+3)(x+2)} x' \alpha_1 + 2 \frac{x+2}{(2x+3)(x-1)} \tau_1\,, \quad x = e_0^2$$

which is solved by

$$\alpha_1 = 2 x^{1/3} (2x+3)^{1/6} (x-1)^{-1} (x+2)^{-1/2} \cdot \int_0^l x^{-1/3} (2x+3)^{-7/6} (x+2)^{3/2} \tau_1 \, dl$$

so that the solubility condition is

$$\oint x^{-1/3} (2x+3)^{-7/6} (x+2)^{3/2} \tau_1 \, dl = 0\,,$$

where the integral is extended over a period of the magnetic axis. A numerical solution for a finite value of ε, obtained in the way described in the next section, is constructed as follows. The zeroth order plane curve (r, φ, z cylindrical coordinates)

$$r(\varphi) = r_0 (1 + r_2 \cos 2\varphi)\,, \quad z = 0\,,$$

is deformed into a spatial curve by

$$z(\varphi) = r_0 (z_2 \sin 2\varphi + z_4 \cos 4\varphi)\,,$$

where z_4 is adjusted in such a way that the solubility condition is satisfied. Figure 1 shows the result for $r_2 = 0.1$, $z_2 = 0.01$, $z_4 = 3.632 \cdot 10^{-3}$.

3. Numerical Study of Toroidal Equilibria with $j = 0$

In order to solve Eqs. (1)–(3) numerically, Eqs. (1)–(2) are considered as equations for α' and e':

$$\alpha' = \frac{e(e^2+1)}{2(2e^2+3)(3e^2+2)(e^2-1)} \cdot [(2e^2+3)\sin\alpha\, f + (3e^2+2)\cos\alpha\, g]\,, \tag{5}$$

$$e' = \frac{e^2}{(2e^2+3)(3e^2+2)} \cdot [(3e^2+2)\sin\alpha\, g - (2e^2+3)\cos\alpha\, f]\,, \tag{6}$$

$$f = -4e\frac{2e^2+1}{e^2+1}\tau\sin\alpha - \cos\alpha \cdot \left[\frac{c_0'}{c_0}\left(e-\frac{2}{e}\right) + 2\frac{\varkappa'}{\varkappa}\left(e+\frac{2}{e}\right)\right] - 8\pi Q_0\,\dot{p}\,c_0^{1/2}e^{1/2}B_1/\varkappa\,,$$

$$g = \frac{4}{e}\frac{e^2+2}{e^2+1}\tau\cos\alpha - \sin\alpha\left[\frac{c_0'}{c_0}\left(\frac{1}{e}-2e\right) + \frac{2\varkappa'}{\varkappa}\left(2e+\frac{1}{e}\right)\right] + 8\pi Q_0\,\dot{p}\,c_0^{1/2}e^{-1/2}B_2/\varkappa\,,$$

so that Eqs. (3), (5), (6) may be solved as four differential equations for the unknowns b_r, b_i, α, e. $\varkappa$, τ, and c_0 are given periodic functions in $[0, L_p]$, where L_p is the period along the magnetic axis. The boundary conditions are periodicity of b_r, b_i, and e and $\alpha(L_p) - \alpha(0) = n\pi$, where n is integer. In the following we set $c_0' \equiv 0$.

As magnetic axes we consider a set of closed curves which are characterized by their number of periods, M, around the torus, positive torsion and curvature, and which approach a helix with $\tau_0 = .9$ and $\varkappa_0 = \sqrt{1-\tau_0^2}$ (where L_p has been normalized to 2π) as $M \to \infty$. Details of this set of magnetic axes are described in the Appendix. $\varkappa$ and τ are chosen to be even functions around $l=0$. As a consequence the boundary conditions $b_r(0)=0$, $\alpha(0)=0$, $b_i'(\pi)=0$, $e'(\pi)=0$ are consistent with periodic solutions having the symmetry: b_r and $\sin\alpha$ odd, b_i, e, α', and $\cos\alpha$ even around 0 and π. Numerically, the equations are solved in such a way that $e(0)$ and $b_i(0)$ are varied in order to make $b_i'(\pi) = e'(\pi) = 0$. Although this procedure does not guarantee that $\alpha(L_p) - \alpha(0) = n\pi$, solutions with this property can be found. In order that the entire class of solutions be parametrized by the single parameter M, we choose a fixed value of $\dot{p}$, $\pi Q_0 \dot{p}/c_0 = -0.56$, which is compatible with the helical limit ($M \to \infty$), in which $e \approx 1.2$ solves Eq. (4). As M is decreased the variations of $\varkappa$ and τ lead to varying e and α. A typical result ($M = 60$) is shown in Figure 2. As M is further decreased the tendency of α to go to $-\pi/2$ at π is increased. At $M=50$ the solutions have changed in character and the elliptical cross-section rotates, as is seen in Figure 3. This type of solution exists continuously down to $M=13$. As an example, $M=20$ is shown in Figure 4. For $M<13$ this branch of solutions does not exist because e approaches 1, which leads to a singularity in Equation (5). However, there exist several other solutions at $M=13$. Apart from the branch found above, which is shown in Fig. 5, there exists a second solution which is also characterized by $e>1$ and $\alpha(\pi) - \alpha(0) = -\pi/2$, which is shown in Figure 6. In addition, there is a third solution with $e<1$ and $\alpha(\pi)-\alpha(0)=0$ (Figure 7). The second branch of solutions has been followed to $M=10$ (see Figure 8).

The equilibrium-β value of these configurations may be estimated in the following way. Since the aspect ratio is a free parameter within the framework of the present theory, a lower bound of the local aspect ratio has to be assumed:

$$A > 1/(\varkappa a)_{\max}\,,$$

where a describes the lateral plasma dimension. The β-value

$$\beta = \frac{\int p\,\mathrm{d}V}{\frac{1}{2}\int B^2\,\mathrm{d}^3\tau}$$

may then be estimated by

$$\beta \approx -\frac{\pi L_p \dot{p}}{c_0^2}\frac{1}{A^2 \varkappa_{\max}^2}$$

where L_p is the period of the magnetic axis. For example, assuming $A>3$, we obtain for the case shown in Fig. 5 $\beta \approx .18$, and the toroidal aspect ratio $|M L_p/(2\pi a_{\max})|$ is about 23 for this case. The magnetic axis and flux surfaces of this configuration are shown in Figure 9.

4. Conclusion

In this paper we have shown that within the framework of the theory used, viz. the expansion about the magnetic axis up to third order in the distance, toroidal equilibria exist whose flux surfaces are elliptical and centered with respect to the magnetic axis. Several extensions of this work are desirable.

Concerning the equilibrium problem, it would be desirable to have a more complete assessment of

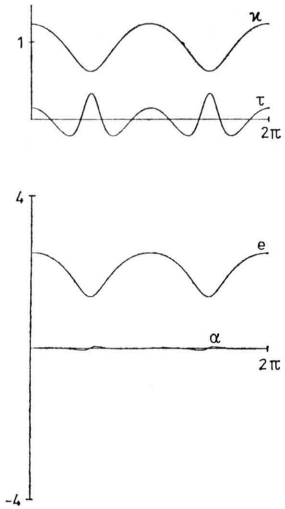
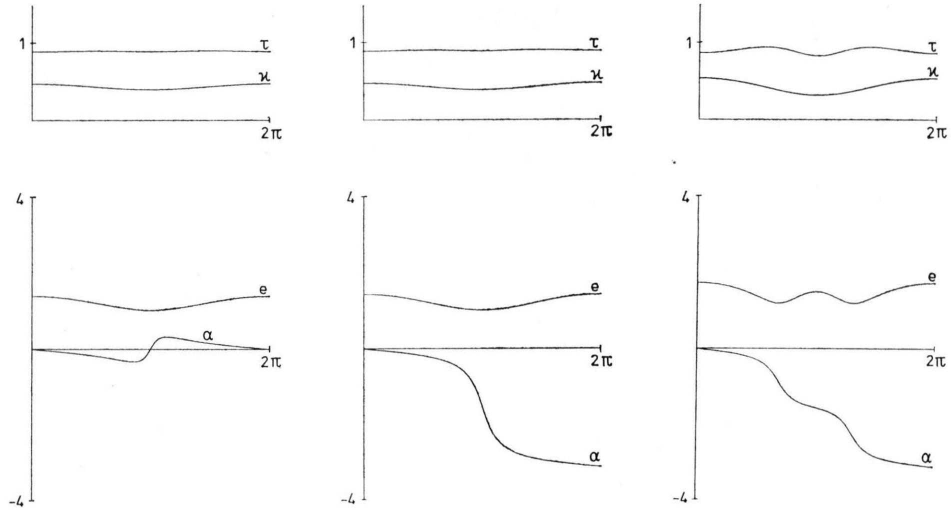
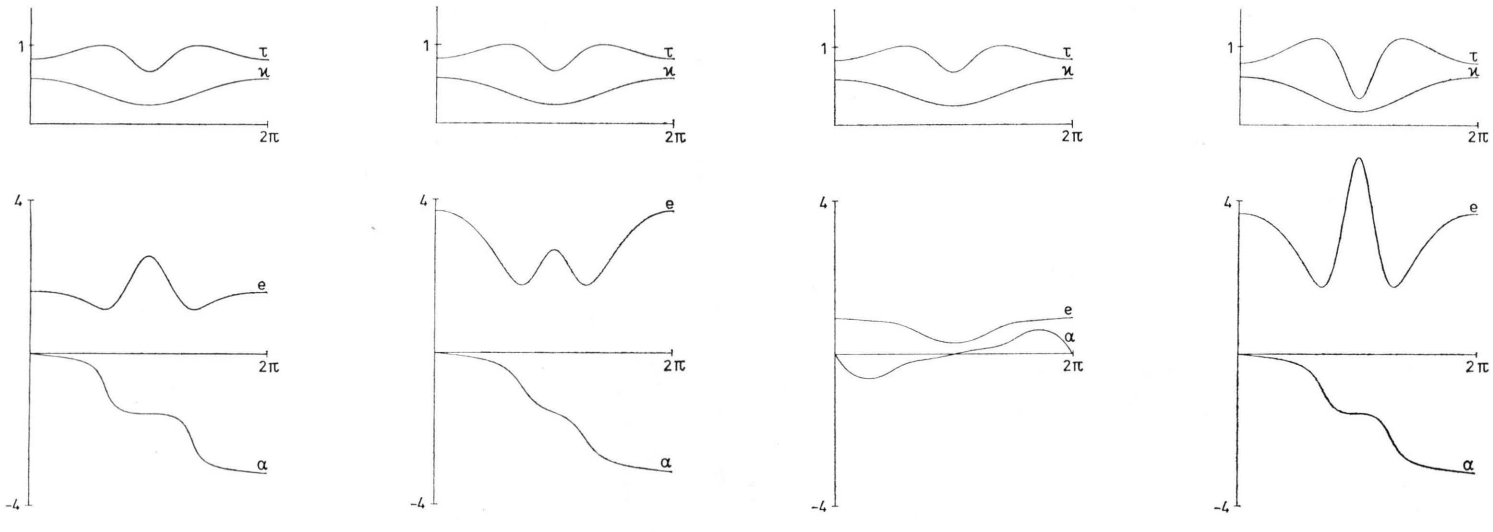


Fig. 1. $\varkappa(\phi)$ and $\tau(\phi)$ for the curve $r = r_0\,(1 + r_2 \cos 2\phi)$, $z(\phi) = r_0(z_2 \sin 2\phi + z_4 \sin 4\phi)$, $r_0 = .990005$, $r_2 = .1$, $z_2 = .01$, $z_4 = 3.632 \cdot 10^{-3}$. The lower part of the figure shows the solutions $e(1)$ and $\alpha(1)$ of Equations (5), (6).



Figs. 2—8. $\varkappa\,(\phi)$ and $\tau\,(\phi)$ for the set of toroidal curves described in the Appendix and various numbers of periods M: Fig. 2 $M = 60$, Fig. 3: $M = 50$, Fig. 4: $M = 20$, Fig. 5: $M = 13$, Fig. 6: $M = 13$, Fig. 7: $M = 13$, Fig. 8: $M = 10$. The lower part of each figure shows the solutions $e\,(1)$ $\alpha\,(1)$ of Equations (5), (6).



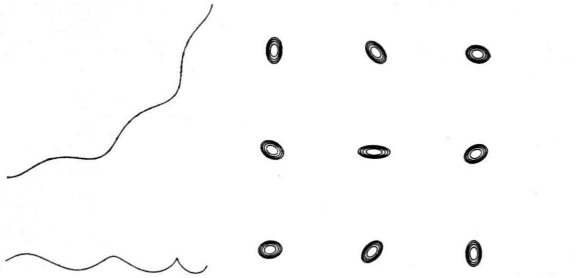


Fig. 9. The left part of the figure shows three periods of the magnetic axis viewed from above (parallel to the torus axis) and from the side (in the torus plane). The right part shows flux surfaces, separated by $L_p/8$, over one period. Shown are the cross-sections of the surfaces with the plane perpendicular to the magnetic axis. The turning angle is given by the rotation of the cross-sections with respect to the intersection of the plane perpendicular to the magnetic axis with the torus plane. The scales for the magnetic surfaces and the magnetic axis are the same.

our result. Calculating the fourth order in the distance from the magnetic axis and three-dimensional numerical calculations could provide such results.

The stability of these equilibria remains to be studied. However, some remarks can already be made. There are completely MHD stable equilibria among these configurations. This follows from the fact that the vacuum configuration considered as the last example in Sect. 2 has a magnetic well. The question how large a β-value can be obtained for a given lower bound on the aspect ratio has not yet been studied. In any case, a β-value obtained in such a way can be improved by admitting stabilizing triangular deformations of the third order flux surfaces.

Appendix

The following gives the description of the set of toroidally closed curves which are taken as magnetic axes in Section 3.

The general formulae

$$\varkappa^2 = \frac{(\dot{\mathbf{r}} \times \ddot{\mathbf{r}})^2}{(\dot{\mathbf{r}}^2)^3}, \quad \tau = -\frac{(\dot{\mathbf{r}} \times \ddot{\mathbf{r}}) \cdot \dddot{\mathbf{r}}}{(\dot{\mathbf{r}} \times \ddot{\mathbf{r}})^2},$$

where $\mathbf{r}$ is the position vector, and the dot the derivative with respect to some parameter increasing with the arclength along the curve, can be evaluated for a curve $r(\varphi)$, $z(\varphi)$ given in cylindrical coordinates r, φ, z to give the results

$$\dot{\mathbf{r}}^2 = r^2 + \dot{z}^2 + \dot{r}^2,$$

$$(\dot{\mathbf{r}} \times \dot{\mathbf{r}})^2 = (r^2 - \ddot{r}\, r + 2\dot{r}^2)^2 + (\ddot{z}\, r - 2\dot{z}\, \dot{r})^2 + (\ddot{r}\, \dot{z} - \ddot{z}\, \dot{r} - \dot{z}\, r)^2,$$

$$(\dot{\mathbf{r}} \times \ddot{\mathbf{r}})\, \dddot{\mathbf{r}} = \dot{z}\,(r^2 + 6\dot{r}^2 - 4\, r\, \ddot{r} + 3\ddot{r}^2 - 2\dot{r}\, \dddot{r}) + \ddot{z}\,(-2\, r\, \dot{r} - 3\dot{r}\, \ddot{r} + r\, \dddot{r}) + \dddot{z}\,(r^2 - r\ddot{r} + 2\dot{r}^2).$$

Here, we consider the set of curves given by

$$r = M\,(r_0 + (r_1/M) \cos M\,\varphi + (r_2/M^2) \cos 2\,M\,\varphi),$$
$$z = r_1 \sin M\,\varphi + (r_2/M) \sin 2\,M\,\varphi.$$

For $r_\nu \sim 0(1)$, $\nu = 0, 1, 2$, and large M, $\varkappa$ and τ of these curves are given by

$$\varkappa = \frac{r_1}{r_0^2 + r_1^2} + \frac{1}{M}\, \frac{r_0(4 r_0\, r_2 + r_0^2 - r_1^2)}{(r_0^2 + r_1^2)^2} \cdot \cos M\,\varphi + O(M^{-2}),$$

$$\tau = \frac{r_0}{r_0^2 + r_1^2} + \frac{1}{M}\left[-\frac{2}{r_1} + \frac{r_0^2}{r_1}\, \frac{(4 r_0\, r_2 + r_0^2 - r_1^2)}{(r_0^2 + r_1^2)^2}\right] \cdot \cos M\,\varphi + O(M^{-2}).$$

Thus, for large M, a helix is approached. Normalization of the helix period to 2π leads to $r_0^2 + r_1^2 = 1$. The choice of r_2 given by

$$4 r_0\, r_2 + r_0^2 - r_1^2 = 2/r_0^2$$

then leads to $\tau = r_0 + O(M^{-2})$. For fixed r_0 (in the numerical study $r_0 = 0.9$ is used) the set of magnetic axes is then described by the single parameter M characterizing the toroidicity.

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